

On the CLT for discrete Fourier transforms of functional time series

Clément Cerovecki¹ Siegfried Hörmann^{1*}

¹ Department of Mathematics, Université libre de Bruxelles CP210, Bd. du Triomphe, B-1050 Brussels, Belgium.

Abstract

Abstract. We consider a strictly stationary functional time series. Our target is to study the weak convergence of the discrete Fourier transforms under sharp conditions. As a side-result we obtain the regular CLT for partial sums under mild assumptions.

Keywords: central limit theorem, functional time series, Fourier transform, periodogram, stationarity

1 Introduction

We consider a time series $(X_t: t \in \mathbb{Z})$ with realizations in some function space H . Then every observation X_t is a random curve $(X_t(u): u \in \mathcal{U})$ with some continuous domain \mathcal{U} . From a technical point of view we require H to be a separable Hilbert space and we call the discrete time process $(X_t: t \in \mathbb{Z})$ a *functional time series (FTS)*.

Functional time series analysis is a branch of the emerging field of *functional data analysis*. It is not unusual that functional data are sequentially sampled and serially correlated by their very nature. A common situation

*Corresponding author. Email: shormann@ulb.ac.be

is that a continuous time process is cut into natural segments, such as days. Then there is not just dependence within the individual curves but also across curves. Nevertheless, growing attention is only given quite recently to this fact. One of the first and most seminal contributors is Bosq (2000), whose monograph is forming the basic theoretical foundation for FTS. A few recent papers devoted to functional time series are Hyndman and Shang (2009), Horváth et al. (2014), Horváth et al. (2015), Aue et al. (2015). Some of the latest publications are related to frequency domain topics for FTS. We refer to Panaretos and Tavakoli (2013a), Panaretos and Tavakoli (2013b), Hörmann et al. (2015a), Hörmann et al. (2015b). In their seminal article on frequency domain methodology for FTS Panaretos and Tavakoli (2013a) have studied, among others, the limiting behavior of the discrete Fourier transform of some FTS (X_t) :

$$S_n(\theta) = \sum_{t=1}^n X_t e^{-it\theta}, \quad \theta \in [-\pi, \pi].$$

This object is of interest to statisticians since it is closely related to the *periodogram* which can, for example, be used to detect some underlying periodic behavior of the time series. (See, e.g., Brockwell and Davis (1991).) Applying such a test for periodicity requires knowledge of the distribution of $S_n(\theta)$. But unless (X_t) is a Gaussian process, the exact distribution is infeasible and then we need to rely on asymptotics. Moreover we notice that with $\theta = 0$ this framework also contains the regular partial sums process, which is without any doubt a crucial building block in many statistical procedures.

For real valued processes asymptotic normality for $S_n(\theta)$ has been obtained under several dependence conditions. Here we only cite the early paper of Walker (1965) who considered linear processes and a recent contribution of Peligrad and Wu (2010), which covers a variety of special cases, including strong mixing sequences. The latter article also contains a more detailed literature survey. For functional data the afore mentioned paper of Panaretos and Tavakoli (2013a) shows that under regularity assumptions $\frac{1}{\sqrt{n}}S_n(\theta)$ converges to a (complex) Gaussian random element whose covariance operator is given by

$$2\pi\mathcal{F}_\theta := \sum_{h \in \mathbb{Z}} C_h e^{-ih\theta}, \tag{1}$$

where $C_h = E[X_h \otimes X_0]$ is the lag h covariance operator of the stationary functional time series. (It is assumed here and henceforth that X_t are

centered by their mean.) The operator \mathcal{F}_θ , which can be shown to be self-adjoint and non-negative definite, is called the *spectral density operator*. In Panaretos and Tavakoli (2013a) it is assumed that $\sum_{h \in \mathbb{Z}} \|C_h\|_\tau < \infty$ (here $\|\cdot\|_\tau$ denotes the Schatten 1-norm) in order to assure convergence of the series defining \mathcal{F}_θ . This assumption is convenient, because it implies immediately that \mathcal{F}_θ is a nuclear operator, i.e. it has a finite trace. This is an important feature when it comes to verifying tightness of $(S_n(\theta)/\sqrt{n})$. Regarding the dependence structure, a cumulant type mixing condition for functional data is used. The nice feature of such mixing conditions is that no specific time series model needs to be imposed. Still, this approach requires to compute and bound functional cumulants of all orders, which is generally not an easy task and necessitates moments of all orders.

One of the main results of this article shows the weak convergence of $S_n(\theta)/\sqrt{n}$ for *purely non-deterministic* processes. More precisely, letting $\mathcal{G}_t = \sigma(X_t, X_{t-1}, \dots)$ and $\mathcal{G}_{-\infty} = \bigcap_{t \geq 0} \mathcal{G}_{-t}$ we impose the following assumption.

Assumption 1. *The process (X_t) is stationary and ergodic and satisfies $E[X_0|\mathcal{G}_{-\infty}] = 0$ a.s.*

We remark that a conditional expectation for random elements in Hilbert spaces as just stated is well defined if $E\|X_0\| < \infty$ (see e.g. Bosq (2000, p.29)). Besides the obligatory existence of second order moments, Assumption 1 will be the only condition needed for the CLT presented below in Theorem 1. This result is an extension of the advances made in Peligrad and Wu (2010) to infinite dimensional data. Since we will not impose any further condition assuring summability of the C_h , a tricky part is the construction and definition of the spectral density operator. Our construction will be an indirect one based on a completeness argument in an appropriate Hilbert space.

In Theorem 2 we will give a result which is slightly less general, but is more useful in applications since it will allow for more explicit constructions of \mathcal{F}_θ . In our Theorem 3 we consider the case $\theta = 0$ and derive the CLT for regular partial sums. These main results are presented in Section 2. In Section 3 we show how the theorems apply in some commonly employed dependence frameworks for functional time series models and compare the required conditions to existing ones in the literature. Proofs are given in Section 4.

2 Main results

We start by introducing further notation and stating the setup precisely. The process (X_t) is defined on some probability space (Ω, \mathcal{A}, P) and takes values in some *separable Hilbert space* H . The space H is equipped with inner product $\langle \cdot, \cdot \rangle$ and the resulting norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. We write $X \in L_H^p(\Omega)$ (short for $X \in L_H^p(\Omega, \mathcal{A}, P)$) to indicate that $E\|X\|^p < \infty$. The space $L_H^p(\Omega)$ is a Banach space and for $p = 2$ again a Hilbert space with inner product $E\langle X, Y \rangle$. We assume throughout that $EX_t = 0$ (the zero element in H) and $X_t \in L_H^2(\Omega)$. Then we denote $C_h = \text{Cov}(X_h, X_0) = E[X_h \otimes X_0]$. That is, $C_h(u) = EX_h \langle u, X_0 \rangle$ for $u \in H$. Expectations or other integrals for elements with values in Banach spaces are understood in the sense of Bochner integrals, see e.g. Mikusiński (1978).

The trace of a self-adjoint and non-negative definite operator $A : H \rightarrow H$ is given by $\text{tr}(A) = \|A\|_{\mathcal{T}} = \sum_{j \geq 1} \langle A(v_j), v_j \rangle$ for some orthonormal basis (v_j) of H . For $X \in L_H^2(\Omega)$ it holds that $\text{tr}(E[X \otimes X]) = E\|X\|^2$. For a general operator A we denote by $\|A\|_{\mathcal{T}}$ the Schatten 1-norm of A and by $\|A\|_{\mathcal{S}}$ its Hilbert-Schmidt norm.

We use $\mathcal{N}_H(\mu, \Sigma)$ to denote a Gaussian element in H with mean μ and covariance operator Σ . Then $X \sim \mathcal{N}_H(\mu, \Sigma)$ if and only if the projection $\langle X, u \rangle$ is normally distributed with mean $\langle \mu, u \rangle$ and variance $\langle \Sigma(u), u \rangle$ for any $u \in H$. Although our observations are assumed to be real, the very definition of $S_n(\theta)$ necessitates to adopt a complex setting. So we will henceforth assume that the Hilbert space $H = H_0 + iH_0$ is complex. Let $EX = \mu = \mu_{\text{Re}} + i\mu_{\text{Im}}$. For $u \in H$ define $\Gamma(u) = E[(X - \mu)\langle u, X - \mu \rangle]$ and $C(u) = E[(X - \mu)\langle X - \mu, u \rangle]$. We say that X is complex Gaussian with mean μ , covariance Γ and relation operator C if

$$\begin{pmatrix} \text{Re}(X) \\ \text{Im}(X) \end{pmatrix} \sim \mathcal{N}_{H_0 \times H_0} \left(\begin{pmatrix} \mu_{\text{Re}} \\ \mu_{\text{Im}} \end{pmatrix}, \frac{1}{2} \begin{bmatrix} \text{Re}(\Gamma + C) & -\text{Im}(\Gamma - C) \\ \text{Im}(\Gamma + C) & \text{Re}(\Gamma - C) \end{bmatrix} \right).$$

Henceforth, we will only need the *circularly-symmetric case*, i.e. when $\mu = 0$ and $C = 0$. Then we write $X \sim \mathcal{CN}_H(0, \Gamma)$. It is straightforward to show that $X \sim \mathcal{CN}_H(0, \Gamma)$, if and only if for any $u \in H$

$$\begin{pmatrix} \text{Re}(\langle X, u \rangle) \\ \text{Im}(\langle X, u \rangle) \end{pmatrix} \sim \mathcal{N}_2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2} \begin{bmatrix} \langle \Gamma(u), u \rangle & 0 \\ 0 & \langle \Gamma(u), u \rangle \end{bmatrix} \right).$$

Finally we recall that a sequence of operators A_n on H is said to converge

in the *weak operator topology* to A if $\langle A_n(u), v \rangle \rightarrow \langle A(u), v \rangle$ for all $u, v \in H$. Short we write $A_n \xrightarrow{w} A$.

Theorem 1. *Let $(X_t: t \in \mathbb{Z})$ be a sequence in $L_H^2(\Omega)$ which satisfies Assumption 1. Then for almost every $\theta \in [-\pi, \pi]$ there exists a linear operator \mathcal{F}_θ , which is self-adjoint and non-negative definite such that*

$$\frac{1}{\sqrt{n}} S_n(\theta) \xrightarrow{d} \mathcal{CN}_H(0, \pi \mathcal{F}_\theta).$$

Moreover we have that

- (I) $\frac{1}{n} E[S_n(\theta) \otimes S_n(\theta)] \xrightarrow{w} 2\pi \mathcal{F}_\theta;$
- (II) $\frac{1}{n} \mathbb{E} \|S_n(\theta)\|_H^2 = \frac{1}{n} \text{tr}(E[S_n(\theta) \otimes S_n(\theta)]) \rightarrow 2\pi \text{tr}(\mathcal{F}_\theta) < \infty;$
- (III) $C_h = \int_{-\pi}^{\pi} \mathcal{F}_\theta e^{ih\theta} d\theta, \quad \forall h \in \mathbb{Z};$
- (IV) *the components of $\frac{1}{\sqrt{n}}(S_n(\theta), S_n(\theta'))$ are asymptotically independent if $\theta \neq \theta'$.*

We call \mathcal{F}_θ the spectral density operator of (X_t) and remark that it is generally not explicitly defined as in (1). In fact, the series in (1) may not be convergent under our mild assumptions. Since

$$2\pi \mathcal{F}_{n;\theta} := \frac{1}{n} E[S_n(\theta) \otimes S_n(\theta)] = \sum_{|h| < n} \left(1 - \frac{|h|}{n}\right) C_h \exp(-ih\theta),$$

relation (I) implies solely that the Cesàro averages of $(C_h \exp(-ih\theta): h \in \mathbb{Z})$ converge (in weak operator topology). Existence of \mathcal{F}_θ will be obtained via a completeness argument in an appropriate Hilbert space.

For practical reasons it is useful to know for which frequencies Theorem 1 holds. For example, $\theta = 0$ is an important special case, but we cannot say if this frequency is part of the exceptional null-set or not. We will see that the critical step in the proof of Theorem 1 is to guarantee existence of the operator \mathcal{F}_θ and to establish the related convergence in (I) and (II). By the extremely mild assumptions we are imposing, we can only assure this for almost every θ . Requiring Assumption 2 below allows us to establish the same result for some fixed frequency θ . To formulate this assumption we first introduce the projection operator $\mathcal{P}_k := E[\cdot | \mathcal{G}_k] - E[\cdot | \mathcal{G}_{k-1}]$, $k \in \mathbb{Z}$.

Assumption 2. *The process (X_t) is stationary and ergodic and for some selected $\theta \in [-\pi, \pi]$ the following properties hold:*

(A1) $\sqrt{2\pi}Z_n(\theta) := \sum_{t=0}^n \mathcal{P}_0(X_t)e^{-it\theta}$ is a Cauchy sequence in $L_H^2(\Omega)$;

(A2) $E\|E[S_n(\theta)|\mathcal{G}_0]\|^2 = o(n)$.

This is our second main result.

Theorem 2. *Let $(X_t: t \in \mathbb{Z})$ be a sequence in $L_H^2(\Omega)$ which satisfies Assumption 2 for some given $\theta \in [-\pi, \pi]$. Then the conclusions of Theorem 1 hold for this particular frequency θ .*

Assumption 2 looks rather technical, but is relatively easy to verify in a variety of models. (See Section 3.) To get some intuition behind these conditions and the general approach we introduce $Z_m^{(k)}(\theta) := \frac{1}{\sqrt{2\pi}} \sum_{t=0}^m \mathcal{P}_k(X_{t+k})e^{-it\theta}$ and write

$$\begin{aligned} S_n(\theta) &= \sum_{k=1}^n \mathcal{P}_k(S_n(\theta)) + E[S_n(\theta)|\mathcal{G}_0] \\ &= \sum_{k=1}^n \sum_{t=k}^n \mathcal{P}_k(X_t)e^{-it\theta} + E[S_n(\theta)|\mathcal{G}_0] \\ &= \sqrt{2\pi} \sum_{k=1}^n Z_{n-k}^{(k)}(\theta)e^{-ik\theta} + E[S_n(\theta)|\mathcal{G}_0]. \end{aligned} \quad (2)$$

The variables $Z_m^{(k)}(\theta)$, $k \geq 1$, are orthogonal and by Assumption (A1) they have a limit $Z^{(k)}(\theta)$ in $L_H^2(\Omega)$. Moreover, it is easy to see that $(Z^{(k)}(\theta))_{k \geq 1}$ is a stationary martingale difference sequence. Together with (A2) this guarantees that $S_n(\theta)$ is in some sense close to $T_n(\theta) := \sum_{k=1}^n Z^{(k)}(\theta)e^{-ik\theta}$. The partial sum $T_n(\theta)$ is more handy when it comes to study the CLT and to compute the covariance operator. In particular,

$$\frac{1}{n} \text{Var}(T_n(\theta)) = \text{Var}(Z^{(0)}(\theta)) =: \mathcal{F}_\theta. \quad (3)$$

We conclude this section with a CLT for regular partial sums. It follows as a corollary of Theorem 2.

Theorem 3. Suppose that $(X_t) \in L_H^2(\Omega)$. If Assumption 2 holds with $\theta = 0$, then

$$\sum_{|h| < n} \left(1 - \frac{|h|}{n}\right) C_h \xrightarrow{w} \mathcal{T},$$

where \mathcal{T} is some non-negative definite, self-adjoint and trace-class operator. Furthermore it holds that $\mathcal{T} = \lim_{n \rightarrow \infty} \text{Var}(\sum_{t=0}^n \mathcal{P}_0(X_t))$ and

$$\frac{1}{\sqrt{n}}(X_1 + \dots + X_n) \xrightarrow{d} \mathcal{N}_H(0, \mathcal{T}).$$

3 Examples

The purpose of this section is to show that our results apply to some commonly used dependence models for functional data. The typical framework we have in mind comprises processes (X_t) which can be represented as Bernoulli shifts, i.e.

$$X_t = f(\varepsilon_t, \varepsilon_{t-1}, \dots) \quad (4)$$

where (ε_t) is a stationary and ergodic sequence of elements in some normed vector space S and $f : S^{\mathbb{N}} \rightarrow H$ is measurable. Then (X_t) is again stationary and ergodic. We remark that in this case we can use in our theorems the filtration $(\mathcal{G}_k : k \in \mathbb{Z})$ with $\mathcal{G}_k = \sigma(\varepsilon_k, \varepsilon_{k-1}, \dots)$. When (ε_t) are i.i.d., then by Kolmogorov's 0-1 law Assumption 1 applies to all such processes. Therefore, only requiring the necessary condition $X_t \in L_H^2(\Omega)$ already implies Theorem 1. But (ε_t) need not necessarily be independent. For example if (ε_t) are *strongly mixing* then the tail sigma algebra $\mathcal{G}_{-\infty}$ is again trivial (see Bradley (2005, p.10)) and hence Assumption 1 still holds.

Representation (4) is very common to many time series models. In particular it applies to the two dependence frameworks we are going to discuss below, namely *linear processes* (possibly with dependent noise) and *$L^2 - m$ -approximable processes*. These two concepts cover most of the functional time series models studied in the literature. Let us beforehand give a simple condition which can replace Assumption 2. Here and in the sequel $\nu_p(X) = (E\|X\|^p)^{1/p}$, $p \geq 1$.

Lemma 1. Assumption 1 and condition **(A3)**: $\sum_{t=0}^{\infty} \nu_2(\mathcal{P}_0(X_t)) < \infty$ together imply Assumption 2 for all $\theta \in [-\pi, \pi]$.

Proof. It is easy to see that **(A3)** implies **(A1)** for all $\theta \in [-\pi, \pi]$.

Consider the decaying sequence of σ -algebras $(\mathcal{G}_{-k} : k \geq 0)$. For any integrable random variable $X \in H$ the process $E[X|\mathcal{G}_{-k}]$ is a reverse martingale with values in H (see e.g. Chatterji (1964)) and consequently the increments $\mathcal{P}_{-k}(X) = E[X|\mathcal{G}_{-k}] - E[X|\mathcal{G}_{-k-1}]$ are orthogonal elements in $L_H^2(\Omega)$. Furthermore, by Assumption 1 it then follows that $E[X|\mathcal{G}_{-k}]$ converges a.s. and in $L_H^2(\Omega)$ to $E[X|\mathcal{G}_{-\infty}] = 0$. Hence, with $X = S_n(\theta)$ we get

$$\begin{aligned} E\|E[S_n(\theta)|\mathcal{G}_0]\|^2 &= \lim_{n \rightarrow \infty} E\|E[S_n(\theta)|\mathcal{G}_0] - E[E[S_n(\theta)|\mathcal{G}_0]|\mathcal{G}_{-n}]\|^2 \\ &= \sum_{j \geq 0} E\|\mathcal{P}_{-j}(E[S_n(\theta)|\mathcal{G}_0])\|^2 = \sum_{j \geq 0} E\|\mathcal{P}_{-j}(S_n(\theta))\|^2. \end{aligned}$$

Therefore we may conclude that

$$\begin{aligned} E\|E[S_n(\theta)|\mathcal{G}_0]\|^2 &\leq \sum_{j=0}^{\infty} E \sum_{s,t=1}^n |\langle \mathcal{P}_{-j}(X_t), \mathcal{P}_{-j}(X_s) \rangle| \\ &= \sum_{j=0}^{\infty} \sum_{s,t=1}^n E |\langle \mathcal{P}_{-j}(X_t), \mathcal{P}_{-j}(X_s) \rangle| \\ &\leq \sum_{s=1}^n \sum_{j=0}^{\infty} \left(\sum_{t=0}^{\infty} \nu_2(\mathcal{P}_0(X_{t+j})) \right) \nu_2(\mathcal{P}_0(X_{s+j})) = o(n). \end{aligned}$$

□

Example 1: Linear processes.

Consider a linear process $X_t = \sum_{k \geq 0} \Psi_k(\varepsilon_{t-k})$ where $(\varepsilon_t)_{t \in \mathbb{Z}}$ are i.i.d. and zero mean in some Hilbert space H' and $\Psi_k : H' \rightarrow H$ are bounded linear operators. We denote by $\|\Psi\|_{\mathcal{L}}$ the operator norm.

Theorem 4. *If $X_t \in L_H^2(\Omega)$ then Theorem 1 holds. If in addition $\kappa := \sum_{k \geq 0} \|\Psi_k\|_{\mathcal{L}} < \infty$ then Theorem 2 applies for all $\theta \in [-\pi, \pi]$. Moreover*

$$\mathcal{F}_\theta = \frac{1}{2\pi} \Psi(\theta) V \Psi(\theta)^*,$$

where $\Psi(\theta) = \sum_{k \geq 0} \Psi_k e^{-ik\theta}$ and $\Psi(\theta)^*$ is its adjoint operator and $V = \text{Var}(\varepsilon_0) := E[\varepsilon_0 \otimes \varepsilon_0]$.

It is easy to see that $\varepsilon_0 \in L^2_{H'}(\Omega)$ implies that $X_t \in L^2_H(\Omega)$. Consequently, our Theorem 4 improves the corresponding result in Panaretos and Tavakoli (2013a), where it is required that $\varepsilon_0 \in L^k_{H'}(\Omega)$ for all $k \geq 1$.

When $\theta = 0$ we recover the ordinary CLT for the partial sums of (X_t) as e.g. proven in Merlevède et al. (1997). While for real-valued linear processes the CLT only requires square summability of coefficients, the latter authors prove that in infinite dimensional Hilbert spaces assuming absolute summability is essentially sharp.

Proof. Noting that $\mathcal{P}_0(X_t) = \Psi_t(\varepsilon_0)$ condition **(A3)** follows immediately from Lemma 1. Also, $\kappa < \infty$ implies $C_h = \sum_{k \geq 0} \Psi_{k+h} V \Psi_k^*$ and this in turn yields that $\sum_{h \in \mathbb{Z}} \|C_h\|_{\mathcal{L}} < \infty$. Therefore \mathcal{F}_θ has representation (1). \square

Example 2: $L^2 - m$ -approximable processes.

Hörmann and Kokoszka (2010) have used the concept of $L^p - m$ -approximability for analyzing dependent functional data. Then a process (X_t) is said to be $L^p - m$ -approximable if X_t has representation (4) with i.i.d. innovations and

$$\sum_{m=1}^{\infty} \nu_p(X_0 - X_0^{(m)}) < \infty,$$

where $X_0^{(m)} = f(\varepsilon_0, \dots, \varepsilon_{0-m+1}, \tilde{\varepsilon}_{-m}, \tilde{\varepsilon}_{-m-1} \dots)$ for some independent copy $(\tilde{\varepsilon}_t)$ of (ε_t) . In Hörmann and Kokoszka (2010) it is shown that this concept applies to many stationary and non-stationary functional time series models, including, for example, functional ARCH. The concept is somewhat related to *near epoch dependence (NED)* often employed in the econometrics literature. See, e.g., Pötscher and Prucha (1997). If this condition holds with $p > 2$, then by a recent result of Berkes et al. (2013) a weak invariance principle for the partial sums process holds. This result has been sharpened by Jirak (2013) who proved the same invariance principle under $p = 2$ and also under a milder coupling condition.

Theorem 5. *Suppose that (X_t) is $L^2 - m$ -approximable. Then Theorem 2 applies for all $\theta \in [-\pi, \pi]$. Moreover, $\sum_{h \in \mathbb{Z}} \|C_h\|_S < \infty$ and therefore*

$$\mathcal{F}_\theta = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} C_h e^{-i\theta h}.$$

Proof. We first note that under $L^2 - m$ -approximability $X_s^{(s)}$ is independent of \mathcal{G}_0 . Hence we get

$$\begin{aligned} E\|\mathcal{P}_0(X_s)\|^2 &= E\|\mathcal{P}_0(X_s - X_s^{(s)})\|^2 \\ &\leq 2E\left(\|E[X_s - X_s^{(s)}|\mathcal{G}_0]\|^2 + \|E[X_s - X_s^{(s)}|\mathcal{G}_{-1}]\|^2\right) \\ &\leq 4E\|X_0 - X_0^{(s)}\|^2 = 4\nu_2^2(X_0 - X_0^{(s)}). \end{aligned}$$

Now apply Lemma 1.

The absolute summability condition on the C_h has been derived in Hörmann et al. (2015a). \square

Example 3: Linear processes with dependent errors

Consider once again a linear process $X_t = \sum_{k \geq 0} \Psi_k(\varepsilon_{t-k})$, where now $(\varepsilon_t)_{t \in \mathbb{Z}}$ is a stationary and ergodic zero mean sequence.

Theorem 6. *Suppose that (X_t) is a linear process satisfying the summability condition $\sum_{k \geq 0} \|\Psi_k\|_{\mathcal{L}} < \infty$. Assume moreover that $(\varepsilon_t)_{t \in \mathbb{Z}}$ satisfies **(A3)**. Then the conclusion of Theorem 5 holds.*

For the regular partial sums process, this result compares to Račkauskas and Suquet (2010) who have studied partial sums of linear processes in Banach spaces. They show that the CLT for the innovations transfers to the linear process under summability of $(\|\Psi_k\|_{\mathcal{L}} : k \geq 0)$.

Proof. We apply Lemma 1. To this end note that

$$\begin{aligned} \sum_{t \geq 0} \nu_2(\mathcal{P}_0(X_t)) &\leq \sum_{t \geq 0} \sum_{k \geq 0} \nu_2(\mathcal{P}_0(\Psi_k(\varepsilon_{t-k}))) \\ &\leq \sum_{k \geq 0} \|\Psi_k\|_{\mathcal{L}} \sum_{t \geq 0} \nu_2(\mathcal{P}_0(\varepsilon_{t-k})) < \infty. \end{aligned}$$

\square

4 Proofs

4.1 Preliminary lemmas

Let $G = L_H^2(\Omega)$ and consider the Hilbert space $L_G^2([-\pi, \pi], \mathcal{B}, \lambda)$, with \mathcal{B} and λ being the Lebesgue measure and the Borel σ -field on $[-\pi, \pi]$, respectively. For simplicity we write $L_G^2([-\pi, \pi])$. This space is equipped with inner product $(V, W) = \int_{-\pi}^{\pi} E\langle V(\theta), W(\theta) \rangle d\theta$ and norm $\|V\| = \sqrt{(V, V)}$.

Lemma 2. *Define $Z_n = Z_n(\theta) := \frac{1}{\sqrt{2\pi}} \sum_{t=0}^n \mathcal{P}_0(X_t) e^{-it\theta}$. Then (Z_n) is a Cauchy sequence in $L_G^2([-\pi, \pi])$, if and only if $\sum_{t \geq 0} E\|\mathcal{P}_{-t}(X_0)\|^2 < \infty$. Moreover, under Assumption 1 the latter summability condition holds.*

We remark that this lemma provides a slightly weaker version of Assumption 2, part 1.

Proof. Using stationarity and the orthogonality of the functions $\theta \mapsto e^{-it\theta}$, $\theta \in [-\pi, \pi]$, $(t \in \mathbb{Z})$ we obtain for $m < n$

$$\begin{aligned} \|Z_n - Z_m\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} E \left\| \sum_{t=m+1}^n \mathcal{P}_0(X_t) e^{-it\theta} \right\|_H^2 d\theta \\ &= \sum_{t=m+1}^n E\|\mathcal{P}_0(X_t)\|^2 = \sum_{t=m+1}^n E\|\mathcal{P}_{-t}(X_0)\|^2. \end{aligned}$$

By the arguments in the proof of Lemma 1 we have

$$\sum_{t \geq 0} E\|\mathcal{P}_{-t}(X_0)\|^2 = \lim_{n \rightarrow \infty} E\|X_0 - E[X_0 | \mathcal{G}_{-n}]\|^2 = E\|X_0\|^2 < \infty.$$

□

It follows under Assumption 1 that there exists an element $Z \in L_G^2([-\pi, \pi])$ with $\|Z_n - Z\| \rightarrow 0$. This in turn has some important implications.

(i) Since

$$\|Z\|^2 = \int_{-\pi}^{\pi} E\|Z(\theta)\|^2 d\theta < \infty,$$

we conclude that $E\|Z(\theta)\|^2 < \infty$ for all $\theta \in M_0 = [-\pi, \pi] \setminus N_0$ where $\lambda(N_0) = 0$. Hence, for all $\theta \in M_0$ the covariance operator $\mathcal{F}_\theta := E[Z(\theta) \otimes Z(\theta)]$ is

well defined, self-adjoint and non-negative definite. The denotation \mathcal{F}_θ is intentional. As we will see later *it is defining the spectral density operator* (compare to (3)). Since $\text{tr}(\mathcal{F}_\theta) = E\|Z(\theta)\|^2$, this operator is trace class. For $\theta \in N_0$ we set $\mathcal{F}_\theta = 0$.

(ii) There exists a sequence (n_k) such that $E\|Z_{n_k}(\theta) - Z(\theta)\|^2 \rightarrow 0$ for all $\theta \in M_1 := [-\pi, \pi] \setminus N_1$, where $\lambda(N_1) = 0$.

(iii) By construction the mapping $\theta \mapsto Z(\theta) \in G$ is measurable, and the mapping $Z(\theta) : G \rightarrow \mathcal{S}$ (the set of Hilbert-Schmidt operators on H), $Z(\theta) \mapsto E[Z(\theta) \otimes Z(\theta)]$, is continuous. Hence, $\theta \rightarrow \mathcal{F}_\theta$ is measurable as a mapping from $[-\pi, \pi]$ to the space \mathcal{S} , which is known to be a separable Hilbert space. Consequently the integral in **(III)** of Theorem 1 is well defined.

The next lemma will be used in the proof of tightness and implies part **(II)** of Theorem 1.

Lemma 3. *Under Assumption 1 we have for all $\theta \in M_2 = [-\pi, \pi] \setminus N_2$ with $\lambda(N_2) = 0$ that*

$$\text{tr}(\mathcal{F}_{n;\theta}) = \frac{1}{2\pi} \sum_{|h|<n} \left(1 - \frac{|h|}{n}\right) E\langle X_h, X_0 \rangle e^{-ih\theta} \rightarrow \text{tr}(\mathcal{F}_\theta) < \infty.$$

Proof. Set $c_h := \int_{-\pi}^{\pi} E\|Z(\theta)\|^2 e^{ih\theta} d\theta$. Using (i) we infer from the Fejér-Lebesgue theorem that

$$\sum_{|h|<n} \left(1 - \frac{|h|}{n}\right) c_h e^{-ih\theta} \rightarrow E\|Z(\theta)\|^2 = \text{tr}(\mathcal{F}_\theta) < \infty \quad \text{for almost all } \theta.$$

We define M_2 as the set of convergence points. We show now that $c_h = E\langle X_h, X_0 \rangle$. Using Lemma 2 and continuity of $\|\cdot\|$, it can be readily shown that

$$c_h = \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} E\|Z_n(\theta)\|^2 e^{ih\theta} d\theta.$$

Without loss of generality assume $h \geq 0$. Using stationarity we deduce

$$\begin{aligned} \int_{-\pi}^{\pi} E\|Z_n(\theta)\|^2 e^{ih\theta} d\theta &= \sum_{t=0}^n \sum_{s=0}^n E\langle \mathcal{P}_0(X_t), \mathcal{P}_0(X_s) \rangle \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(t-s-h)\theta} d\theta \\ &= \sum_{t=h}^n E\langle \mathcal{P}_0(X_t), \mathcal{P}_0(X_{t-h}) \rangle = \sum_{t=h}^n E\langle \mathcal{P}_{-t}(X_0), \mathcal{P}_{-t}(X_{-h}) \rangle. \end{aligned}$$

Since the terms $\mathcal{P}_{-t}(X_0)$ and $\mathcal{P}_{-s}(X_{-h})$ are orthogonal in $L_H^2(\Omega)$ for $s \neq t$, it follows that

$$\int_{-\pi}^{\pi} E \|Z_n(\theta)\|^2 e^{ih\theta} d\theta = E \left\langle \sum_{t=h}^n \mathcal{P}_{-t}(X_0), \sum_{s=h}^n \mathcal{P}_{-s}(X_{-h}) \right\rangle.$$

We recall that $(E[X_0|\mathcal{G}_{-t}]: t \geq 0)$ is a reverse martingale, it follows by Assumption 1 that

$$\sum_{t=h}^n \mathcal{P}_{-t}(X_0) = E[X_0|\mathcal{G}_{-h}] - E[X_0|\mathcal{G}_{-(n+1)}] \xrightarrow{L_H^2(\Omega)} E[X_0|\mathcal{G}_{-h}]. \quad (5)$$

Similarly, $\sum_{s=h}^n \mathcal{P}_{-s}(X_{-h}) \xrightarrow{L_H^2(\Omega)} E[X_{-h}|\mathcal{G}_{-h}] = X_{-h}$. And hence, by continuity of the inner product, $c_h = E\langle E[X_0|\mathcal{G}_{-h}], X_{-h} \rangle = E\langle X_h, X_0 \rangle$. \square

Lemma 4. *The operators \mathcal{F}_θ define the spectral density operators of (X_t) at frequency θ . This is*

$$C_h = \int_{-\pi}^{\pi} \mathcal{F}_\theta e^{ih\theta} d\theta, \quad \forall h \in \mathbb{Z}.$$

Proof. We have seen in (iii) that the mapping $\theta \mapsto \mathcal{F}_\theta$ is measurable. The integrand is valued in the separable Hilbert space \mathcal{S} . Since $\int_{-\pi}^{\pi} \|\mathcal{F}_\theta\|_S d\theta \leq \int_{-\pi}^{\pi} \text{tr}(\mathcal{F}_\theta) d\theta < \infty$ we know that \mathcal{F}_θ is strongly integrable and hence we can define (in the sense of a Bochner integral) $I = \int_{-\pi}^{\pi} \mathcal{F}_\theta e^{ih\theta} d\theta$. Let $u, v \in H$. Since Bochner integrals are interchangeable with bounded linear operators we obtain

$$\begin{aligned} \langle I(v), u \rangle &= \int_{-\pi}^{\pi} \langle \mathcal{F}_\theta(v), u \rangle e^{ih\theta} d\theta = \int_{-\pi}^{\pi} E \langle Z(\theta), u \rangle \overline{\langle Z(\theta), v \rangle}_H e^{ih\theta} d\theta \\ &= \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} E \langle Z_n(\theta), u \rangle \overline{\langle Z_n(\theta), v \rangle}_H e^{ih\theta} d\theta. \end{aligned}$$

The last equality can be deduced from $\|Z_n - Z\| \rightarrow 0$. Assume now with loss of generality that $h \geq 0$. Similar arguments as in Lemma 3 lead to

$$\langle I(v), u \rangle = \lim_{n \rightarrow \infty} E \left[\left\langle \sum_{t=0}^{n-h} \mathcal{P}_{-t}(X_h), u \right\rangle \left\langle v, \sum_{s=0}^{n-h} \mathcal{P}_{-s}(X_0) \right\rangle \right].$$

From (5) it follows that

$$\langle I(v), u \rangle = E[\langle E[X_h | \mathcal{G}_0], u \rangle \langle v, X_0 \rangle] = \langle C_h(v), u \rangle.$$

Since u and v are arbitrary in H , we can infer that $I = C_h$. \square

We show next that the projections $\langle S_n(\theta), u \rangle / \sqrt{n}$, $u \in H$, converge weakly to $\langle S_0(\theta), u \rangle$, where $S_0(\theta) \sim \mathcal{CN}_H(0, \pi \mathcal{F}_\theta)$ is the limiting complex Gaussian element and where \mathcal{F}_θ is defined in (i). The first step towards this result is given by the following proposition.

Proposition 1. *Under Assumption 1 there exists for all $u \in H$ a set $\tilde{N} \subset [-\pi, \pi]$ of Lebesgue measure 0, such that on $\tilde{M} = [-\pi, \pi] \setminus \tilde{N}$ the following holds:*

- (a) $\lim_{n \rightarrow \infty} \text{Var}(\langle S_n(\theta), u \rangle) / n = \lim_{n \rightarrow \infty} 2\pi \langle \mathcal{F}_{n;\theta}(u), u \rangle = 2\pi \langle \mathcal{F}_\theta(u), u \rangle;$
- (b) $(\text{Re}(\langle S_n(\theta), u \rangle), \text{Im}(\langle S_n(\theta), u \rangle)) / \sqrt{n} \xrightarrow{d} \mathcal{N}_2(0, \pi \langle \mathcal{F}_\theta(u), u \rangle \times I_2);$

Proof. We first show that there exists for all $u \in H$ a set $N_u \subset [-\pi, \pi]$ with $\lambda(N_u) = 0$, such that on $M_u = [-\pi, \pi] \setminus N_u$ (a) and (b) hold.

Let $u \in H$ and let $\mathcal{G}_k(u)$ be the filtration of the process $(\langle X_t, u \rangle : t \in \mathbb{Z})$. From the results in Peligrad and Wu Peligrad and Wu (2010) we obtain that $\text{Var}(\langle S_n(\theta), u \rangle) / n \rightarrow 2\pi f^u(\theta)$ for some function $f^u(\theta)$ which is finite on M_u . More precisely, slightly adapting the proofs of Lemmas 4.1. and 4.2. in their article we obtain that the $L^2(\Omega)$ limit

$$Z^u(\theta) := \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \sum_{t=0}^n \mathcal{P}_0(\langle X_t, u \rangle) e^{-it\theta} = \lim_{n \rightarrow \infty} \langle Z_n(\theta), u \rangle$$

exists on M_u and that $f^u(\theta) = \text{Var}(Z^u(\theta))$. (Directly using their arguments would require to use the projection operator $\mathcal{P}_0^u(\cdot) = E[\cdot | \mathcal{G}_0(u)] - E[\cdot | \mathcal{G}_{-1}(u)]$.) We assume without loss of generality that $M = M_0 \cap M_1$ is a subset of M_u , otherwise replace M_u by $M_u \cap M$. We now determine $f^u(\theta)$. By result (ii) of Section 4.1 it follows that $E(\langle Z_{n_k}(\theta), u \rangle - \langle Z(\theta), u \rangle)^2 \rightarrow 0$ for every $u \in H$ and all $\theta \in M_1$. Hence, by result (i) in the same section, we get

$$\text{Var}(\langle Z_{n_k}(\theta), u \rangle) \rightarrow \langle \mathcal{F}_\theta(u), u \rangle < \infty,$$

for all $\theta \in M$ and all $u \in H$, which implies on M_u the relation $f^u(\theta) = \langle \mathcal{F}_\theta(u), u \rangle < \infty$. This shows part (a) on M_u .

We have $E[\langle X_t, u \rangle | \mathcal{G}_{-\infty}] = \langle E[X_t | \mathcal{G}_{-\infty}], u \rangle$ and by Assumption 1 this is equal to zero. The tower property of conditional expectations implies $E[\langle X_t, u \rangle | \mathcal{G}_{-\infty}(u)] = 0$ and hence on M_u **(b)** directly follows from Peligrad and Wu (2010). Let us note that their CLT result is stated for real valued time series, but this requirement is not needed. Hence we can apply it for the time series $(\langle X_t, u \rangle : t \geq 1)$ which takes values in \mathbb{C} when $u \in H$.

It remains to prove that for all u in H we can find a common exceptional set of Lebesgue measure 0. To this end let H' be a dense and countable subset of H . We set $\widetilde{M} = \cap_{u' \in H'} M_{u'}$. Then $[-\pi, \pi] \setminus \widetilde{M}$ has Lebesgue measure 0. Furthermore, for all $u' \in H'$ and $\theta \in \widetilde{M}$ **(a)** and **(b)** hold. The objective is now to extend this result to all $u \in H$. For **(a)** we observe that

$$\begin{aligned} & |\langle \mathcal{F}_{n;\theta}(u), u \rangle - \langle \mathcal{F}_\theta(u), u \rangle| \\ & \leq |\langle \mathcal{F}_{n;\theta}(u), u \rangle - \langle \mathcal{F}_{n;\theta}(u'), u' \rangle| + |\langle \mathcal{F}_\theta(u'), u' \rangle - \langle \mathcal{F}_\theta(u), u \rangle| \\ & \quad + |\langle \mathcal{F}_{n;\theta}(u'), u' \rangle - \langle \mathcal{F}_\theta(u'), u' \rangle| \\ & \leq [\text{tr}(\mathcal{F}_{n;\theta}) + \text{tr}(\mathcal{F}_\theta)] \times [(\|u\| + \|u'\|) \times \|u - u'\|] \\ & \quad + |\langle \mathcal{F}_{n;\theta}(u'), u' \rangle - \langle \mathcal{F}_\theta(u'), u' \rangle|. \end{aligned}$$

Since we can assume without loss of generality that $M_2 \subset \widetilde{M}$, it follows that for all $\theta \in \widetilde{M}$

$$\limsup_{n \rightarrow \infty} |\langle \mathcal{F}_{n;\theta}(u), u \rangle - \langle \mathcal{F}_\theta(u), u \rangle| \leq 4\varepsilon(\|u\| + 1) \text{tr}(\mathcal{F}_\theta),$$

if $\|u - u'\| \leq \varepsilon \leq 1$. Since ε can be chosen arbitrarily small result **(a)** follows.

The proof of part **(b)** follows along similar lines of arguments. Just compare the characteristic functions of the real and complex part of $\langle S_n(\theta), u \rangle$ to the corresponding normal ones. \square

4.2 Tightness

Lemma 5. Consider sequences $(p_j^{(n)} : j \geq 1)$, $n \geq 0$, with the following properties: (a) $p_j^{(n)} \geq 0$ for all j, n ; (b) $\lim_n p_j^{(n)} = p_j^{(0)}$; (c) $\sum_{j \geq 1} p_j^{(0)} = p < \infty$; (d) $\lim_n \sum_{j \geq 1} p_j^{(n)} = p$; (e) $\sum_{j \geq 1} p_j^{(n)} < \infty$ for all $n \geq 1$. Then

$$\lim_{m \rightarrow \infty} \sup_n \sum_{j > m} p_j^{(n)} = 0.$$

Proof. Fix an $\varepsilon > 0$. We have to show that for $m \geq m(\varepsilon)$ we have $\sum_{j \geq m} p_j^{(n)} < \varepsilon$ for all $n \geq 1$.

By (c) we can choose $m_1 = m_1(\varepsilon)$ such that $\sum_{j \geq m} p_j^{(0)} < \varepsilon/3$ for all $m \geq m_1$. Furthermore, by (b) we can choose a large enough $n_1 = n_1(\varepsilon)$ such that for all $n \geq n_1$ we have $|\sum_{j=1}^{m_1} p_j^{(0)} - \sum_{j=1}^{m_1} p_j^{(n)}| < \varepsilon/3$. Next, by possibly further enlarging n_1 we deduce from (c) and (d) that $|\sum_{j \geq 1} p_j^{(n)} - \sum_{j \geq 1} p_j^{(0)}| < \varepsilon/3$. Consequently, for $n \geq n_1$, we have

$$\begin{aligned} \sum_{j > m_1} p_j^{(n)} &= \sum_{j \geq 1} p_j^{(n)} - \sum_{j=1}^{m_1} p_j^{(n)} \\ &\leq |\sum_{j \geq 1} p_j^{(n)} - \sum_{j \geq 1} p_j^{(0)}| + |\sum_{j \geq 1} p_j^{(0)} - \sum_{j=1}^{m_1} p_j^{(0)}| + |\sum_{j=1}^{m_1} p_j^{(0)} - \sum_{j=1}^{m_1} p_j^{(n)}| \\ &< \varepsilon. \end{aligned}$$

Because of (a) this bound is still valid for all $m \geq m_1$. For the n_1 just chosen, we can find an $m_2 = m_2(\varepsilon)$, such that $\sum_{j > m_2} p_j^{(n)} < \varepsilon$ for all $n \leq n_1$. This is because of (d) and (e) we know that $\sup_{n \geq 1} \sum_{j \geq 1} p_j^{(n)} < \infty$. And again, because of (a) we know also that $\sum_{j > m} p_j^{(n)} < \varepsilon$ for all $m \geq m_2$ and $n \leq n_1$. Hence, set $m(\varepsilon) = \max\{m_1, m_2\}$. \square

Lemma 6. *Take some ONB (v_j) of H . Lemma 5 applies with $p_j^{(n)} = \langle \mathcal{F}_{n;\theta}(v_j), v_j \rangle$, $p_j^{(0)} = \langle \mathcal{F}_\theta(v_j), v_j \rangle$ for all $\theta \in \widetilde{M}$.*

Proof. We can assume that $v_j \in H'$ for all $j \geq 1$. Relation (a) is trivial. Relation (b) follows from part (a) of Proposition 1. Relation (c) holds because \mathcal{F}_θ is nuclear on \widetilde{M} . And similarly relation (e) holds because apparently $\mathcal{F}_{n;\theta}$ is nuclear for any n . Finally note that (d) can be reformulated as $\text{tr}(\mathcal{F}_{n;\theta}) \rightarrow \text{tr}(\mathcal{F}_\theta)$. By Lemma 3 this holds for almost all $\theta \in M_2 \subset \widetilde{M}$. \square

Lemma 7. *Under Assumptions 1 the sequence $(S_n(\theta)/\sqrt{n}: n \geq 1)$ is tight for all $\theta \in \widetilde{M}$.*

Proof. Let $\varepsilon > 0$. We consider the sequences $0 < l_k \nearrow \infty$ and $0 < N_k \nearrow \infty$ and define

$$K = \bigcap_{k=1}^{\infty} \left\{ x \in H : \sum_{j > N_k} |\langle v_j, x \rangle|^2 \leq \frac{1}{l_k} \right\}.$$

Just as in Bosq (2000) (p. 52) we can see that it is a compact subset of H . We now have that

$$\begin{aligned} P(S_n(\theta)/\sqrt{n} \in K) &\geq 1 - \sum_{k=1}^{\infty} l_k \sum_{j>N_k} E|\langle S_n(\theta)/\sqrt{n}, v_j \rangle|^2 \\ &= 1 - \sum_{k=1}^{\infty} 2\pi l_k \sum_{j>N_k} \langle \mathcal{F}_{n;\theta}(v_j), v_j \rangle, \end{aligned}$$

where we used the σ -subadditivity and the Markov inequality. By Lemma 6 we know that

$$\sup_n \sum_{j \geq m} \langle \mathcal{F}_{n;\theta}(v_j), v_j \rangle \rightarrow 0 \quad (m \rightarrow \infty).$$

Therefore, for any $\varepsilon > 0$, we can choose increasing sequences (l_k) and (N_k) such that

$$2\pi l_k \sum_{j>N_k} \langle \mathcal{F}_{n;\theta}(v_j), v_j \rangle \leq \varepsilon 2^{-k}.$$

□

4.3 Proofs of Theorems 1 and 2

Proof of Theorem 1. Parts **(II)** and **(III)** of Theorem 1 follow directly from Lemmas 3 and 4. Part **(I)** can be deduced from the polarization identity for self-adjoint operators Γ

$$\begin{aligned} \langle \Gamma(x), y \rangle &= \frac{1}{4} [\langle \Gamma(x+y), x+y \rangle - \langle \Gamma(x-y), x-y \rangle \\ &\quad + i\langle \Gamma(x+iy), x+iy \rangle - i\langle \Gamma(x-iy), x-iy \rangle], \end{aligned}$$

and part **(a)** of Proposition 1. Next, the asymptotic normality of $S_n(\theta)\sqrt{n}$ for all $\theta \in \widetilde{M}$ follows from the corresponding convergence of the projections (Proposition 1, part **(b)**) and the tightness shown in Lemma 7. Finally, the asymptotic independence relation **(IV)** can be obtained by verifying it for the projections $\langle S_n(\theta), u \rangle/\sqrt{n}$ and $\langle S_n(\theta'), u \rangle/\sqrt{n}$. For this we can refer to Peligrad and Wu (2010). □

Proof of Theorem 2. Let $\theta \in [-\pi, \pi]$ be such that Assumption 2 is satisfied. Due to relation (2) we have that

$$\begin{aligned} \frac{E\|S_n(\theta)\|^2}{n} &= \frac{2\pi}{n} \sum_{k=1}^n E\|Z_{n-k}^{(0)}(\theta)\|^2 + \frac{1}{n} E\|E[S_n(\theta)|\mathcal{G}_0]\|^2 \\ &= 2\pi E\|Z^{(0)}(\theta)\|^2 + o(1), \end{aligned}$$

where we used both **(A1)** and **(A2)**. Since $\text{tr}(\mathcal{F}_\theta) = E\|Z^{(0)}(\theta)\|^2$ we conclude that **(II)** holds for the fixed θ . Note that condition **(A1)** just provides a stronger version of Lemma 2. There we only get the limit of $Z_n (= Z_n^{(0)})$ in $L_G^2([-\pi, \pi])$, whereas now we get it pointwise by Assumption **(A1)**. In other words, we got rid of the exceptional sets N_1 and N_2 in Lemmas 2 and 3. The proof of Lemma 4 is unchanged, hence **(III)** follows.

Proposition 1 can be proven even more easily for a given $\theta \in [-\pi, \pi]$. For part **(a)** we project (2) onto $u \in H$ and deduce just as above that

$$\frac{E|\langle S_n(\theta), u \rangle|^2}{n} = 2\pi E|\langle Z^{(0)}(\theta), u \rangle|^2 + o(1).$$

This also shows **(I)** by the polarization identity. Part **(b)** can be shown by the same martingale approximation as in the proof of Theorem 2.1. of Peligrad and Wu (2010). Alternatively, one may directly apply Theorem 2 in Wu (2004).

Finally the tightness of $S_n(\theta)/\sqrt{n}$ can be shown using the same proof as in Lemma 7 since it is now clear that relations (a)-(e) of Lemma 5 are satisfied for the particular θ . \square

Acknowledgement

This research was supported by the Communauté française de Belgique—Actions de Recherche Concertées (2010–2015) and from the Interuniversity Attraction Poles Programme (IAP-network P7/06), Belgian Science Policy Office. Clément Cervecki acknowledges support from the F.R.S.-FNRS Fond de la Recherche Scientifique, Rue d’Egmont 5, B-1000 Bruxelles.

References

- AUE, A., DUBART NORINHO, D., AND HÖRMANN, S. (2015). *On the prediction of stationary functional time series*. Journal of the American Statistical Association, 110, 509, 378–392.
- BERKES, I., HORVÁTH, L. AND RICE, G. (2013). *Weak invariance principles for sums of dependent random functions*. Stochastic Processes and their Applications, 123, 385–403.
- BOSQ, D. (2000). *Linear Processes in Function Spaces*. Springer, New York.
- BROCKWELL, P.J. AND DAVIS, R.A. (1991). *Time Series: Theory and Methods*. Springer, New York.
- BRADLEY, R.C. (2005). *Basic Properties of Strong Mixing Conditions. A Survey and Some Open Questions*. Probability Surveys Vol. 2, 107–144.
- CHATTERJI, S.D. (1964). *A note on the convergence of Banach-space valued martingales*. Mathematische Annalen, Volume 153, Issue 2, pp 142–149.
- HÖRMANN, S., KIDZIŃSKI, L. AND HALLIN, M. (2015). *Dynamic Functional Principal Component*. Journal of the Royal Statistical Society: Series B, 77, 319–348.
- HÖRMANN, S., KIDZIŃSKI, L. AND KOKOSZKA, P. *Estimation in functional lagged regression*. Journal of Time Series Analysis. (Forthcoming. DOI: 10.1111/jtsa.12114.)
- HÖRMANN, S., KOKOSZKA, P. (2010). *Weakly dependent functional data*. The Annals of Statistics, 38, 1845–1884.
- HORVÁTH, L., KOKOSZKA, P., AND RICE, G. (2014). *Testing stationarity of functional time series*. Journal of Econometrics, 179, 66–82.
- HORVÁTH, L., RICE, G., AND WHIPPLE, S. (2014). *Adaptive bandwidth selection in the estimation of the long run covariance of functional time series*. Computational Statistics and Data Analysis. Forthcoming.
- HYNDMAN, R.J. AND SHANG, H.L. (2009). *Forecasting functional time series*. Journal of the Korean Statistical Society, 38, No. 3, 199–211.

- JIRAK, M. (2013). *On weak invariance principles for sums of dependent random functionals*. Statistics and Probability Letters, 83, 2291–2296.
- MERLEVÈDE, F., PELIGRAD, M. AND UTEV, S. (1997). *Sharp Conditions for the CLT of Linear Processes in a Hilbert Space*. Journal of Theoretical Probability, 10, No. 3, 681–693.
- MIKUSIŃSKI, J. (1978). *The Bochner Integral*. Birkhauser, Basel.
- PANARETOS, V.M. AND TAVAKOLI, S. (2013). *Fourier analysis of stationary time series in function spaces*. The Annals of Statistics, 41, No. 2, 568–603.
- PANARETOS, V.M. AND TAVAKOLI, S. (2013). *Cramér–Karhunen–Loève representation and harmonic principal component analysis of functional time series*. Stochastic Processes and their Applications, 123, 2779–2807.
- PELIGRAD, M. AND WU, W.B. (2010). *Central limit theorem for Fourier transforms of stationary processes*. The Annals of Probability, 38, No. 5, 2009–2022.
- PÖTSCHER, B.M. AND PRUCHA, I.R. (1997). *Dynamic Nonlinear Econometric Models*. Springer.
- RAČKAUSKAS, A. AND SUQUET, CH. (2010). *On limit theorems for Banach-space-valued linear processes*. Lithuanian Mathematical Journal, 50, No. 1, 71–87.
- WALKER, A.M. (1965). *Some asymptotic results for the periodogram of a stationary time series*. Journal of the Australian Mathematical Society, 5, No. 1, 107–128.
- WU, W.B. (2004). *Fourier transforms of stationary processes*. Proceeding of the American society, Vol 133, Number 1, Pages 285–293.